Problem 1.

2CLIQUE:

*Instance:* An undirected graph $G$, positive integer $K$;

*Query:* Does $G$ have two disjoint cliques each of size $K$?

We give a polynomial-time reduction from CLIQUE to 2CLIQUE:

Given a graph $G$ and integer $k$, construct graph $G'$ by making two disjoint copies of $G$, then taking $G'$ to be their union.

More formally, if $G = (V, E)$, take $G' = (V', E')$ where $V' = \{[i, v] : i \in \{0, 1\}, v \in V\}$ and $E' = \{([i, v], [i, w]) : i \in \{0, 1\}, (v, w) \in E\}$.

The reduction is then $f(G, k) = (G', k)$.

The reduction is clearly polynomial time.

To prove that it is correct, we show that $G$ has a clique of size $k$ if and only if $G'$ has two disjoint cliques of size $k$. This is fairly obvious, but we'll prove it anyway.

$(\Rightarrow)$ Suppose $G$ has a clique $C$ of size $k$. Then for $i = 1, 2$ let $C_i = \{[i, v] : v \in C\}$. Then $C_1$ and $C_2$ are disjoint, and each is a clique in $G'$. So $G'$ has two disjoint cliques of size $k$.

$(\Leftarrow)$ Suppose $G'$ has two disjoint cliques $C_1$ and $C_2$ of size $k$. Consider the clique $C_1$. Since there are no edges of the form $([0, v], [1, w])$ in $G'$, either all of the vertices of $C_1$ are of the form $[0, v]$ or they are all of the form $[1, v]$. That is, $C_1$ is contained entirely in one of the two copies of $G$. Define $C = \{v : [0, v] \in C_1 \lor [1, v] \in C_1\}$. Then $C$ is a clique in $G$ (prove it if you like), and $C$ has size $k$.

Problem 2.

VISIT-EDGES:

*Instance:* An undirected graph $G = (V, E)$, set of edges $F \subseteq E$;

*Query:* Is there a cycle in $G$ that traverses each edge in $F$?

Note: above we mean a *simple* cycle, that is a cycle that does not visit the same vertex twice.

We give a reduction from HAMILTONIAN CYCLE. Given a graph $G = (V, E)$, construct the bipartite graph $G' = (V', E')$, where:

$$V' = \{[i, v] : i \in \{0, 1\}, v \in V\}$$

$$E' = A \cup B,$$

where

$$A = \{([i, v], [j, w]) : (v, w) \in E, i \neq j\},$$

$$B = \{([0, v], [1, v]) : v \in V\}.$$

In words, we are making two copies $[0, v]$ and $[1, v]$ of every vertex $v$ in the original graph. We add an edge between each pair of vertices like that, then for each edge $(u, w)$ in the original graph we add two edges $([0, u], [1, w])$ and $([0, w], [1, u])$. 

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The reduction is \( f(G) = (G', B) \).

Clearly the reduction is polynomial time. Next we prove it is correct — that \( G \) has a simple cycle that visits all of its vertices if and only if \( G' \) has a simple cycle that crosses every edge in \( A \).

(\( \Rightarrow \)) Suppose \( G \) has a simple cycle \( C \) that visits all its vertices. Name the vertices along cycle \( C \) as \( v_1, v_2, \ldots, v_n, v_1 \) in the order they are visited by \( C \).

Define cycle \( C' \) to visit the vertices (in \( G' \)) as follows:

\[
[0, v_1], [1, v_1], [0, v_2], [1, v_2], [0, v_3], [1, v_3], \ldots, [0, v_n], [1, v_n], [0, v_1].
\]

Then, because \( C \) is a simple cycle in \( G \) that visits all the vertices in \( V \), \( C' \) is a simple cycle in \( G' \) (verify this!) that visits all the edges in \( A \).

(\( \Leftarrow \)) Suppose \( G' \) has a simple cycle \( C' \) that uses all the edges in \( A \). Since every vertex in \( G' \) is on an edge in \( A \), the cycle \( C' \) must be a Hamiltonian cycle in \( G' \). Since each vertex in \( G' \) touches exactly one edge in \( A \), the edges traversed by the cycle \( C' \) must alternate between edges in \( A \) and edges not in \( A \) (verify!). Also, all of the edges in \( E' \) connect a vertex \([i, v]\) to a vertex \([j, w]\) where \( i \neq j \). Thus, vertices \( C' \) visits, in order, can be named as

\[
[0, v_1], [1, v_1], [0, v_2], [1, v_2], \ldots, [0, v_n], [1, v_n], [0, v_1]
\]

where the \( v_i \)'s are distinct and, for each \( i \), \([0, v_i], [1, v_i]\) is an edge in \( A \) and \((v_i, v_{i+1})\) is an edge in \( E \) (the original graph). Thus, the cycle \( C = (v_1, v_2, \ldots, v_n, v_1) \) is a Hamiltonian cycle in \( G \).

Problem 3.

SET-SPLITTING:

**Instance:** A finite set \( S \) and a collection \( C \) of finite subsets of \( S \);

**Query:** Can the elements of \( S \) be colored with two colors, say red and green, so that no set \( X \in C \) has all elements colored with the same color?

As an example, suppose that \( S = \{1, 2, 3, 4, 5, 6\} \) and

\[
C = \{\{1, 2\}, \{3, 4, 5\}, \{2, 3, 6\}, \{1, 4, 6\}, \{2, 5\}\}.
\]

If 1, 3, 5 are green, and 2, 4, 6 are red, then each set in \( C \) has elements of two different colors.

In the following instance:

\[
C = \{\{1, 2\}, \{3, 4, 5\}, \{2, 3, 6\}, \{1, 4\}, \{2, 5\}, \{1, 3\}, \{5, 6\}\},
\]

there is no good coloring (why?).

To prove SET-SPLITTING is NP-Complete, we need to show (1) SET-SPLITTING is in NP and (2) SET-SPLITTING is NP-Hard.

To see (1) is easy because, given a coloring, it is easy to verify in poly time that no set is monochromatic.

To prove that SET-SPLITTING is NP-hard, we reduce 3SAT to it.

Let \( \phi \) be a 3CNF formula with variable set \( V \).

Construct the following instance of SET-SPLITTING. The set of elements \( S \) is \( \{F\} \cup V \cup \{\bar{X} : X \in V\} \). Here \( F \) is a new element not related to any variable. Each other element corresponds to a variable or its negation.
Build the collection of sets $C$ as follows. For each variable $X$ in $\phi$, construct a set $S_X = \{X, \overline{X}\}$. For each clause $c$ (e.g. $X \lor Y \lor \overline{Z}$), construct a set $S_c = \{X, Y, \overline{Z}, F\}$. Here $F$ is a new element not related to any variable. (There is only one such element $F$ — it is the same across all sets built for clauses.)

So, the reduction is $f(\phi) = (S, C)$ where $S$ and $C$ are as described above.

Clearly the reduction is polynomial time.

To finish we prove that $\phi$ is satisfiable if and only if $(S, C)$ can be colored so that no set is monochromatic.

$(\Rightarrow)$ Suppose $\phi$ is satisfiable. Fix some satisfying assignment.

Consider the following coloring of the elements in $S$. Color the element $F$ 'red'. For each variable $X$ that is assigned 'false', color the elements $X$ and $\overline{X}$ 'red' and 'green', respectively. For each variable $X$ that is assigned 'true', color the elements $X$ and $\overline{X}$ 'green' and 'red', respectively.

As long as the assignment was satisfying, this coloring makes no set monochromatic. For each variable $X$, the set $S_X = \{X, \overline{X}\}$ has one red and one green element. For each clause $c$, the set $S_c$ has at least one red element ($F$) and, because some literal in the clause has a value of true, $S_c$ has at least one 'green' element.

Thus, $(S, C) \in \text{SET-SPLITTING}$. 

$(\Leftarrow)$ Suppose $(S, C) \in \text{SET-SPLITTING}$. Fix some coloring of $S$ with two colors such that every set has at least one element of both colors.

Consider the following assignment to the variables of $\phi$. For each variable $X$, assign it 'true' if its color differs from that of the element $F$. Assign $X$ 'false' if its color is the same as that of the element $F$.

Then each clause $c$ in $\phi$ is satisfied, because the set $S_c$ has at least one element $X$ or $\overline{X}$ that is colored differently than $F$.

Thus, $\phi \in \text{SAT}$.

Collection: I will collect the homeworks in class on Tuesday. If you can’t turn it in in class, slip it under my office door (by no later than 8AM).